# Linear independence results for the values of divisor functions series (Analytic Number Theory and Related Areas)



Luca, Florian; Tachiya, Yohei

数理解析研究所講究録
第2014巻 2017年 138-150



# Linear independence results for the values of divisor functions series

FLORIAN LUCA
School of Mathematics
University of the Witwatersrand

YOHEI TACHIYA
Graduate School of Science and Technology,
Hirosaki University

**Abstract**

Let $\{a_1(n)\}_{n\geq 1}$ be a purely periodic sequence of nonnegative integers, not identically zero, and $\{a_\ell(n)\}_{n\geq 1}$ $(\ell=2,3,\ldots)$ be the sequences defined inductively by $a_\ell(n):=\sum_{d|n}a_{\ell-1}(d)$. Then, for an arbitrary integer $q$ $(|q|>1)$, the numbers

$$1 \quad \text{and} \quad \sum_{n=1}^{\infty}a_\ell(n)q^{-n} \quad (\ell=2,3,\ldots)$$

are linearly independent over $\mathbb{Q}$. In particular, the numbers

$$1 \quad \text{and} \quad \sum_{n=1}^{\infty}d_\ell(n)q^{-n} \quad (\ell=2,3,\ldots)$$

are linearly independent over $\mathbb{Q}$, where $d_\ell(n)$ are generalized divisor functions.

**2010** Mathematics Subject Classification: 11J72, 11A25.
**Key words:** irrationality, linear independence, divisor function.

## 1 Introduction

For an integer $\ell\geq 1$, we define the arithmetic function $d_\ell(n)$ as the number of ordered factorization of $n$ into exactly $\ell$ factors, namely, the number of $\ell$-tuples of positive integers $(d_1,\ldots,d_\ell)$ with $n=d_1\cdots d_\ell$. For example, $d_1(n)=1$ $(n\geq 1)$ and $d_2(n)$ denotes the number of positive divisors of $n$. The arithmetic function $d_\ell(n)$ is sometimes called the *generalized divisor function.* For each $\ell\geq 1$, the functions $d_\ell(n)$ is multiplicative. Indeed, the function $d_\ell(n)$ is given by the Dirichlet convolution

$$d_\ell(n)=(d_1*d_{\ell-1})(n)=\sum_{m|n}d_{\ell-1}(m) \quad (n\geq 1),$$

where the sum is taken over all positive divisors $m$ of $n$. This relation implies that the function $d_\ell(n)$ can be obtained from the Dirichlet series expression of the $\ell$th power of Riemann zeta function

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$:

$$\zeta(s)^\ell = \sum_{n=1}^{\infty} \frac{d_\ell(n)}{n^s} \quad (\mathrm{Re}\, s > 1).$$

Let $\{a_1(n)\}_{n\geq 1}$ be a sequence of integers and $\{a_\ell(n)\}_{n\geq 1}$ ($\ell = 1, 2, \dots$) be the sequences defined inductively by

$$a_\ell(n) := \sum_{m|n} a_{\ell-1}(m) \quad (n \geq 1). \tag{1}$$

For example, the functions $d_\ell(n)$ ($\ell = 1, 2, \dots$) are generated from the unit function $a_1(n) = 1$ ($n \geq 1$). Consider the power series

$$f_\ell(z) := \sum_{n=1}^{\infty} a_\ell(n) z^n \quad (\ell = 1, 2, \dots). \tag{2}$$

If $\{a_1(n)\}_{n\geq 1}$ is a periodic sequence, then the functions (2) converge in $|z| < 1$, since $a_\ell(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$ (see Lemma 3). Furthermore, the function $f_1(z)$ is a rational function in $z$ in the region $|z| < 1$ and the functions $f_\ell(z)$ ($\ell = 2, 3, \dots$) are expressed by (1) as Lambert series

$$f_\ell(z) = \sum_{n=1}^{\infty} \frac{a_{\ell-1}(n) z^n}{1 - z^n} \quad (|z| < 1).$$

In 1948, Erdős [2] gave the irrationality of

$$\sum_{n=1}^{\infty} d_2(n) q^{-n} = \sum_{n=1}^{\infty} \frac{1}{q^n - 1}$$

for any integer $q > 1$ by showing that the $q$-adic expansion contains any arbitrary long string of zeros without being identically zero from some point on. In [3], we generalized Erdős' result as follows:

**Theorem A** ([3, Theorem 1.1]) *Let $\{a_1(n)\}_{n\geq 1}$ be a purely periodic sequence of integers, not identically zero, and $\{a_2(n)\}_{n\geq 1}$ be a sequence defined by (1). Then the value*

$$f_2(q^{-1}) = \sum_{n=1}^{\infty} a_2(n) q^{-n} = \sum_{n=1}^{\infty} \frac{a_1(n)}{q^n - 1}$$

*is irrational for any integer $q$ ($|q| > 1$).*

In this paper, under the nonnegativity condition on $\{a_1(n)\}_{n\geq 1}$, we generalize Theorem A by proving the linear independence result for the values of the power series (2).

Throughout this paper, let $q$ be an integer with $|q| > 1$.

**Theorem 1.** *Let $\{a_1(n)\}_{\geq 1}$ be a purely periodic sequence of nonnegative integers, not identically zero, and $\{a_\ell(n)\}_{n\geq 1}$ ($\ell = 2, 3, \dots, m$) be sequences defined by (1). Then the $m$ numbers*

$$1 \quad \text{and} \quad f_\ell(q^{-1}) = \sum_{n=1}^{\infty} a_\ell(n) q^{-n} \quad (\ell = 2, 3, \dots, m) \tag{3}$$

*are linearly independent over $\mathbb{Q}$.*

**Example 1.** *The $m$ numbers*

$$1 \quad \text{and} \quad \sum_{n=1}^{\infty} d_\ell(n) q^{-n} = \sum_{n=1}^{\infty} \frac{d_{\ell-1}(n)}{q^n - 1} \quad (\ell = 2, 3, \ldots, m)$$

*are linearly independent over $\mathbb{Q}$.*

**Example 2.** *Let $\{a_1(n)\}_{n\geq 1}$ be the sequence defined by $a_1(2k-1) = 1$ and $a_1(2k) = 0$ for $k \geq 1$, and $\{a_\ell(n)\}_{n\geq 1}$ ($\ell = 2, 3, \ldots$) be the sequences defined by (1). Then the numbers*

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{q^{2n-1} - 1}, \quad \sum_{n=1}^{\infty} \frac{a_2(n)}{q^n - 1}, \ldots \quad , \sum_{n=1}^{\infty} \frac{a_\ell(n)}{q^n - 1}, \ldots$$

*are linearly independent over $\mathbb{Q}$.*

**Remark 1.** It should be noted that the proof of Theorem 1 is elementary in the sense that we do not need a deep result about primes in arithmetic progressions by Alford, Granville, and Pomerance [1], as in our previous paper [3] for example. This simplification is due to the nonnegativity condition on $\{a_1(n)\}_{n\geq 1}$.

## 2 Lemmas

In this section, we derive an upper bound for the summatory function of $a_\ell(n)$ over arithmetic progressions (Lemma 4). Let $d(n) := d_2(n)$ be the number of positive divisors of $n$.

**Lemma 1.** *Let $k \geq 1$ be an integer. Then we have for $N \geq 1$*

$$\sum_{n=1}^{N} \frac{d(n)^k}{n} \leq (1 + \log N)^{2^k}.$$

*Proof.* Since $d(m\ell) \leq d(m)d(\ell)$ for any integers $m$ and $\ell$, we have

$$\begin{aligned}
\sum_{n=1}^{N} \frac{d(n)^k}{n} &= \sum_{n=1}^{N} \sum_{m|n} \frac{d(n)^{k-1}}{n} = \sum_{m=1}^{N} \left( \sum_{\substack{1\leq n\leq N \\ m|n}} \frac{d(n)^{k-1}}{n} \right) \\
&= \sum_{m=1}^{N} \sum_{\ell=1}^{\lfloor \frac{N}{m} \rfloor} \frac{d(m\ell)^{k-1}}{m\ell} \leq \sum_{m=1}^{N} \frac{d(m)^{k-1}}{m} \sum_{\ell=1}^{\lfloor \frac{N}{m} \rfloor} \frac{d(\ell)^{k-1}}{\ell} \\
&\leq \left( \sum_{m=1}^{N} \frac{d(m)^{k-1}}{m} \right)^2.
\end{aligned}$$

Hence we obtain inductively

$$\sum_{n=1}^{N} \frac{d(n)^k}{n} \leq \left( \sum_{m=1}^{N} \frac{1}{m} \right)^{2^k} \leq (1 + \log N)^{2^k}.$$

□

**Lemma 2.** *Let $k \geq 1$ be an integer. Let $A \geq 1$ and $B$ be coprime integers with $-A < B < 2A$. Then we have*

$$\sum_{i=1}^{N} d(Ai+B)^k \leq 2^{2^{k+1}} N(1+\log N)^{2^k}$$

*for every integer $N$ with $N \geq (2A)^{2^k-1}$.*

*Proof.* Since $\sqrt{AN+B} \leq \sqrt{2AN} \leq N$,

$$\begin{aligned}
\sum_{i=1}^{N} d(Ai+B)^k &= \sum_{i=1}^{N} d(Ai+B)^{k-1} \sum_{m|Ai+B} 1 \\
&\leq \sum_{i=1}^{N} d(Ai+B)^{k-1} \left( 2 \sum_{\substack{m|Ai+B \\ m\leq\sqrt{Ai+B}}} 1 \right) \\
&\leq 2 \sum_{m=1}^{N} \sum_{\substack{1\leq i\leq N \\ m|Ai+B}} d(Ai+B)^{k-1}. \qquad (4)
\end{aligned}$$

Suppose that $m$ divides $Ai+B$. Since $A$ and $B$ are coprime, so are $A$ and $m$. Hence there exists a unique integer $r_m$ in the range $-m+1 \leq r_m \leq 0$ such that $i \equiv -A^{-1}B \equiv r_m \pmod{m}$. Let $i = mj + r_m$. Then there exist at most $\lfloor \frac{N+m-1}{m} \rfloor \leq \lfloor \frac{N}{m} \rfloor + 1$ numbers $j$ such that $1 \leq i \leq N$, so that

$$\begin{aligned}
\sum_{\substack{1\leq i\leq N \\ m|Ai+B}} d(Ai+B)^{k-1} &\leq \sum_{j=1}^{\lfloor \frac{N}{m} \rfloor+1} d(Amj + Ar_m + B)^{k-1} \\
&\leq d(m)^{k-1} \sum_{j=1}^{\lfloor \frac{N}{m} \rfloor+1} d\left(Aj + \frac{Ar_m+B}{m}\right)^{k-1}. \qquad (5)
\end{aligned}$$

Thus, for $k=1$, we obtain by (4) and (5)

$$\begin{aligned}
\sum_{i=1}^{N} d(Ai+B) &\leq 2\sum_{m=1}^{N} \left( \left\lfloor \frac{N}{m} \right\rfloor + 1 \right) \leq 4N \sum_{m=1}^{N} \frac{1}{m} \\
&\leq 4N(1+\log N).
\end{aligned}$$

We continue the proof of Lemma 2 by induction on $k$. By the above argument, the claim holds for $k=1$. Let $k \geq 2$ and assume the lemma is true for $k-1$. In the right hand side of (5), the integers $A$ and $\frac{Ar_m+B}{m}$ are coprime with

$$-A < \frac{Ar_m+B}{m} < 2A.$$

Furthermore by the assumption $N \geq (2A)^{2^k-1}$,

$$\left\lfloor \frac{N}{m} \right\rfloor + 1 \geq \frac{N}{m} \geq \frac{N}{\sqrt{AN+B}} \geq \frac{N}{\sqrt{2AN}} \geq (2A)^{2^{k-1}-1}.$$

Hence, we obtain, by the induction hypothesis

$$\begin{aligned}
\sum_{j=1}^{\lfloor \frac{N}{m} \rfloor + 1} d\left(Aj + \frac{Ar_m + B}{m}\right)^{k-1} &\leq 2^{2^k} \left(\frac{2N}{m}\right) \left(1 + \log\left(\frac{2N}{m}\right)\right)^{2^{k-1}} \\
&\leq 2^{2^k + 2^{k-1} + 1} \left(\frac{N}{m}\right) (1 + \log N)^{2^{k-1}}. \qquad (6)
\end{aligned}$$

Therefore by Lemma 1 together with (4), (5), and (6),

$$\begin{aligned}
\sum_{i=1}^{N} d(Ai+B)^k &\leq 2^{2^{k+1}} N(1+\log N)^{2^{k-1}} \sum_{m=1}^{N} \frac{d(m)^{k-1}}{m} \\
&\leq 2^{2^{k+1}} N(1+\log N)^{2^k}.
\end{aligned}$$

This completes the proof of Lemma 2. □

Let $\{a_1(n)\}_{n\geq 1}$ be a purely periodic sequence of nonnegative integers, not identically zero, and $\{a_\ell(n)\}_{n\geq 1}$ ($\ell = 2, 3, \dots$) be sequences defined by (1). Define $a := \max\{a_1(n) : n \geq 1\} > 0$.

**Lemma 3.** *For each $\ell = 1, 2, \dots$, we have*

$$a_\ell(n) \leq a \cdot d(n)^{\ell-1} \quad (n \geq 1).$$

*Proof.* The assertion is trivial for $\ell = 1$ and we have by the induction on $\ell$

$$\begin{aligned}
a_\ell(n) &= \sum_{m|n} a_{\ell-1}(m) \leq \sum_{m|n} a \cdot d(m)^{\ell-2} \\
&\leq a \cdot d(n)^{\ell-2} \sum_{m|n} 1 \\
&= a \cdot d(n)^{\ell-1}.
\end{aligned}$$

□

**Lemma 4.** *Let $A$ and $B$ be coprime integers with $-A < B < 2A$. For each $\ell = 1, 2, \dots$, the inequality*

$$\sum_{i=1}^{N} a_\ell(Ai+B) \leq 2^{2^\ell} a N(1+\log N)^{2^{\ell-1}}$$

*holds for any integer $N$ with $N \geq (2A)^{2^{\ell-1}-1}$.*

*Proof.* This follows immediately from Lemmas 2 and 3. □

**Lemma 5.** *Let $s \geq 1$ be a period length of $\{a_1(n)\}_{\geq 1}$. Suppose that the positive integer $n$ has the form $n = m\prod_i p_i^{e_i}$, where $p_i$ are distinct prime numbers with $p_i \equiv 1 \pmod{s}$ and coprime with $m$. Then, for each $\ell = 1, 2, \ldots$, the function $a_\ell(n)$ is expressed as*

$$a_\ell(n) = a_\ell(m)\prod_i \binom{e_i + \ell - 1}{\ell - 1}. \tag{7}$$

*Proof.* The claim holds for $\ell = 1$, since $n \equiv m \pmod{s}$ and $\{a_1(n)\}_{\geq 1}$ is periodic sequence with a period length $s$. Let $\ell \geq 2$ and assume that (7) holds for $\ell - 1$. Then we have by the induction hypothesis

$$\begin{aligned}
a_\ell(n) &= \sum_{d|n} a_{\ell-1}(d) = \sum_{d_1|m\prod_{i=1}^{k-1} p_i^{e_i}} \left( \sum_{d_2|p_k^{e_k}} a_{\ell-1}(d_1 d_2) \right) \\
&= \sum_{d_1|m\prod_{i=1}^{k-1} p_i^{e_i}} \sum_{j=0}^{e_k} a_{\ell-1}(d_1 p_k^j) \\
&= \sum_{d_1|m\prod_{i=1}^{k-1} p_i^{e_i}} \sum_{j=0}^{e_k} a_{\ell-1}(d_1) \binom{j + \ell - 2}{\ell - 2} \\
&= \binom{e_k + \ell - 1}{\ell - 1} \sum_{d_1|m\prod_{i=1}^{k-1} p_i^{e_i}} a_{\ell-1}(d_1),
\end{aligned}$$

where we used the equality

$$\sum_{j=0}^{e_k} \binom{j + \ell - 2}{\ell - 2} = \binom{e_k + \ell - 1}{\ell - 1}.$$

Repeating this process, or applying induction over the values of $k = 1, 2, \ldots$, we obtain

$$\begin{aligned}
a_\ell(n) &= \left( \sum_{d|m} a_{\ell-1}(d) \right) \prod_i \binom{e_i + \ell - 1}{\ell - 1} \\
&= a_\ell(m) \prod_i \binom{e_i + \ell - 1}{\ell - 1},
\end{aligned}$$

which gives the desired result. □

Applying Lemma 5 to the function $d_\ell(n)$, we have the formula

$$d_\ell(n) = \prod_{p|n} \binom{v_p(n) + \ell - 1}{\ell - 1},$$

where $v_p(n)$ is the exponent of $p$ in the prime factorization of $n$ (cf. [4, Theorem 7.5]).

## 3 Preliminaries

Let $m \geq 2$ be an integer and $\{\theta(n)\}_{n\geq 1}$ a sequence defined by the linear combination of $\{a_\ell(n)\}_{n\geq 1}$ $(\ell = 2, 3, \ldots, m)$ over $\mathbb{Z}$:

$$\theta(n) := \sum_{\ell=2}^{m} b_\ell a_\ell(n) \quad (b_\ell \in \mathbb{Z}). \tag{8}$$

Let $p_1$ be the least prime with $p \equiv 1 \pmod{s}$ and $p_1, p_2, \ldots$ be increasing sequence of all the primes which are congruent to 1 modulo $s$, where $s \geq 1$ be a period length of $\{a_1(n)\}_{n\geq 1}$. We choose a sufficiently large integer $k$ with $k > p_1$ and put

$$t_k := \frac{k(k+1)}{2}, \qquad r_k := t_k + 1.$$

We denote $q_1, q_2, \ldots, q_{t_{2k}}$ by the first $t_{2k}$ odd prime numbers satisfying $q_i \equiv 1 \pmod{s}$ and all greater than $4k^3$. Let $L := m!$ and $q$ be an integer with $|q| > 1$. Then, by the Chinese Remainder Theorem, there exists an integer $B_k$ satisfying the congruences

$$\begin{cases} B_k - k + 1 \equiv q_1^{|q|L-1} & \left(\bmod \;\; q_1^{|q|L}\right), \\ B_k - k + 2 \equiv (q_2 q_3)^{|q|L-1} & \left(\bmod \;\; (q_2 q_3)^{|q|L}\right), \\ \vdots & \vdots \\ B_k - 1 \equiv (q_{r_{k-2}} \cdots q_{t_{k-1}})^{|q|L-1} & \left(\bmod \;\; (q_{r_{k-2}} \cdots q_{t_{k-1}})^{|q|L}\right), \\ B_k + 1 \equiv (q_{r_k} \cdots q_{t_{k+1}})^{|q|L-1} & \left(\bmod \;\; (q_{r_k} \cdots q_{t_{k+1}})^{|q|L}\right), \\ \vdots & \vdots \\ B_k + k \equiv (q_{r_{2k-1}} \cdots q_{t_{2k}})^{|q|L-1} & \left(\bmod \;\; (q_{r_{2k-1}} \cdots q_{t_{2k}})^{|q|L}\right), \end{cases} \tag{9}$$

which furthermore is unique under the additional inequality

$$1 \leq B_k \leq A_k,$$

where

$$A_k := \prod_{\substack{i=1 \\ i \neq r_{k-1}, \ldots, t_k}}^{t_{2k}} q_i^{|q|L}.$$

In what follows, let $c_1, c_2, \ldots$ be positive constants which may depend on $q$, $m$, and the function $\{a_1(n)\}_{n\geq 1}$ (in fact, only on $s$ and $a := \max\{a_1(n) : 1 \leq n \leq s\}$) but are independent of $k$. Since the $n$th prime $p_n$ in the arithmetic progression $p_i \equiv 1 \pmod{s}$ satisfies the inequality

$$p_n \leq 2sn \log n$$

for large $n$, we have

$$B_k \leq A_k \leq \prod_{i=1}^{t_{2k}} p_{i+4k^3}^{|q|L} \leq e^{c_1 k^2 \log k}. \tag{10}$$

Let $N_k := 2^{k^3}$ and

$$S(k) := \{u_{k,i} := A_k i + B_k \mid i = 1, \ldots, N_k\}.$$

We put $p := p_1$ and choose a positive integer $\nu$ with $p < |q|^\nu$. Let $h \geq 1$ be the least integer with $a(h) = a$. Define the subsets of $S(k)$:

$$T_1 = T_1(k) := \{u_{k,i} \in S(k) \mid u_{k,i} \equiv 0 \pmod{hp^{\lfloor \frac{k}{\nu+1} \rfloor}}\},$$

$$T_\ell = T_\ell(k) := \{u_{k,i} \in S(k) \mid a_\ell(u_{k,i}) < 2^{2^\ell} a p^{\frac{k}{\nu}} (1 + \log N_k)^{2^{\ell-1}}\}$$

for each $\ell = 2, 3, \ldots, m$, and put

$$T = T(k) := \bigcap_{\ell=1}^{m} T_\ell.$$

**Lemma 6.** *There exists an integer $i_k$ $(1 \leq i_k \leq N_k)$ such that*

$$u_{k,i_k} = A_k i_k + B_k \in T,$$

*such that*

$$\sum_{n=1}^{2mk^3} |\theta(u_{k,i_k} + n + k)| \leq p^{\frac{k}{\nu}}.$$

*Proof.* First, we estimate lower bounds for the number of elements in each $T_\ell$. Since $1 \leq h \leq s$ and

$$p_1 = p < k < 4k^3 < q_i < A_k,$$

the integer $A_k$ is coprime with $hp$. Hence, we have

$$\sharp T_1 \geq \left\lfloor \frac{N_k}{hp^{\lfloor \frac{k}{\nu+1} \rfloor}} \right\rfloor \geq \frac{N_k}{hp^{\frac{k}{\nu+1}}} - 1, \tag{11}$$

where $\sharp T_1$ denotes the number of elements in the set $T_1$. On the other hand, for each $\ell = 2, 3, \ldots, m$, we have, by Lemma 4,

$$\sum_{i=1}^{N} a_\ell(Ai + B) \leq 2^{2^\ell} a N (1 + \log N)^{2^{\ell-1}}$$

for any coprime integers $A$ and $B$ with $-A < B < 2A$, if $N \geq (2A)^{2^{\ell-1}-1}$. Hence, putting $A := A_k$, $B := B_k$, and $N := N_k$, we get for each $\ell = 2, 3, \ldots, m$,

$$\begin{aligned} 2^{2^\ell} a N_k (1 + \log N_k)^{2^{\ell-1}} &\geq \sum_{i=1}^{N_k} a_\ell(u_{k,i}) \geq \sum_{\substack{i=1 \\ u_{k,i} \notin T_\ell}}^{N_k} a_\ell(u_{k,i}) \\ &\geq (N_k - \sharp T_\ell) \cdot 2^{2^\ell} a p^{\frac{k}{\nu}} (1 + \log N_k)^{2^{\ell-1}}, \end{aligned}$$

which implies

$$\sharp T_\ell \geq \left(1 - \frac{1}{p^{\frac{k}{\nu}}}\right) N_k.$$

Thus, we have

$$\sharp(T_2 \cap T_3) \geq \sharp T_2 + \sharp T_3 - N_k \geq \left(1 - \frac{2}{p^{\frac{k}{\nu}}}\right) N_k,$$

and inductively

$$\sharp(\cap_{\ell=2}^{m} T_\ell) \geq \left(1 - \frac{m-1}{p^{\frac{k}{\nu}}}\right) N_k. \tag{12}$$

Therefore, we obtain, by (11) and (12),

$$\sharp T = \sharp(\cap_{\ell=1}^{m} T_\ell) \geq \left(\frac{N_k}{hp^{\frac{k}{\nu+1}}} - 1\right) - \frac{m-1}{p^{\frac{k}{\nu}}} N_k \geq \frac{N_k}{2hp^{\frac{k}{\nu+1}}}. \tag{13}$$

Define

$$\beta_k := \sum_{i=1}^{N_k} \sum_{n=1}^{2mk^3} |\theta(u_{k,i} + n + k)|.$$

By Lemma 4 with $A := A_k$, $B := B_k + n + k$ and $N := N_k$, we have the following upper bound which is uniform in $n \in \{1, 2, \ldots, 2mk^3\}$:

$$\begin{aligned}
\beta_k &\leq M \sum_{n=1}^{2mk^3} \sum_{\ell=2}^{m} \sum_{i=1}^{N_k} a_\ell(A_k i + B_k + n + k) \\
&\leq M \sum_{n=1}^{2mk^3} \sum_{\ell=2}^{m} 2^{2^\ell} a N_k (1 + \log N_k)^{2^{\ell-1}} \\
&\leq 2am^2 M k^3 \cdot 2^{2^m} N_k (1 + \log N_k)^{2^{m-1}} \\
&\leq c_2 k^{3 \cdot 2^m} N_k,
\end{aligned} \tag{14}$$

where $M := \max_s |b_s|$. Thus, putting

$$\alpha_k := \min_{\substack{i=1,2,\ldots,N_k \\ u_{k,i} \in T}} \left(\sum_{n=1}^{2mk^3} |\theta(u_{k,i} + n + k)|\right),$$

we obtain, by (13) and (14),

$$\begin{aligned}
\alpha_k \frac{N_k}{2hp^{\frac{k}{\nu+1}}} &\leq \sum_{\substack{i=1 \\ u_{k,i} \in T}}^{N_k} \left(\sum_{n=1}^{2mk^3} |\theta(u_{k,i} + n + k)|\right) \\
&\leq \beta_k \\
&\leq c_2 k^{3 \cdot 2^m} N_k,
\end{aligned}$$

which implies that $\alpha_k \leq p^{\frac{k}{\nu}}$ for all sufficiently large $k$. $\square$

Let $i_k$ be as in Lemma 6 and put $u_k := u_{k,i_k} \in T$.

**Lemma 7.** *For sufficiently large $k$, we have*

$$\left|\sum_{n=1}^{\infty} \frac{\theta(u_k+n+k)}{q^n}\right| \leq 2p^{\frac{k}{\nu}}.$$

*Proof.* By (10), we have

$$u_k + 2mk^3 + k = A_k i_k + B_k + 2mk^3 + k \leq 2^{2k^3},$$

and hence, by Lemma 3,

$$\begin{aligned} |\theta(u_k+2mk^3+n+k)| &\leq M\sum_{\ell=2}^{m} a_\ell(u_k+2mk^3+n+k) \\ &\leq aM\sum_{\ell=2}^{m} d(u_k+2mk^3+n+k)^{\ell-1} \\ &\leq aM\sum_{\ell=2}^{m} (u_k+2mk^3+n+k)^{\ell-1} \\ &\leq 2^{2mk^3} mMn^m. \end{aligned} \tag{15}$$

Thus, we get, by Lemma 6 together with (15),

$$\begin{aligned} \left|\sum_{n=1}^{\infty} \frac{\theta(u_k+n+k)}{q^n}\right| &\leq \sum_{n=1}^{2mk^3} |\theta(u_k+n+k)| + \sum_{n=2mk^3+1}^{\infty} \frac{|\theta(u_k+n+k)|}{|q|^n} \\ &\leq p^{\frac{k}{\nu}} + \sum_{n=1}^{\infty} \frac{|\theta(u_k+2mk^3+n+k)|}{|q|^{2mk^3+n}} \\ &\leq p^{\frac{k}{\nu}} + mM\left(\frac{2}{|q|}\right)^{2mk^3} \sum_{n=1}^{\infty} \frac{n^m}{|q|^n} \\ &\leq 2p^{\frac{k}{\nu}}. \end{aligned}$$

□

**Lemma 8.** *Suppose that the infinite series*

$$b_1 := \sum_{n=1}^{\infty} \frac{\theta(n)}{q^n} \tag{16}$$

*is an integer. Then $\theta(u_k) = 0$ holds for every large $k$.*

*Proof.* We have, by (9),

$$u_k + j = A_k i_k + B_k + j = m_{k,j} \prod_{i=r_{j+k-1}}^{t_{j+k}} q_i^{|q|L-1}$$

for each nonzero integer $j = -k+1, \ldots, k$, where $m_{k,j}$ is a positive integer coprime with all the primes $q_i$ for $i \in \{r_{j+k-1}, \ldots, t_{j+k}\}$. By (7), we get

$$\begin{aligned} a_\ell(u_k + j) &= a_\ell(m_{k,j}) \binom{|q|L + \ell - 1}{\ell - 1}^{t_{j+k} - r_{j+k-1} + 1} \\ &= \mu_{k,j,\ell} |q|^{k+j} \end{aligned}$$

for $\ell = 2, 3, \ldots, m$, where $\mu_{k,j,\ell}$ is an integer because $(\ell - 1)! \mid L$. Hence,

$$\theta(u_k + j) = \sum_{\ell=2}^{m} b_\ell a_\ell(u_k + j) \equiv 0 \pmod{|q|^{k+j}},$$

for each $j = -k+1, \ldots, k$ $(j \neq 0)$ and, by (16),

$$\begin{aligned} b_1 &= \sum_{n=1}^{u_k - k} \frac{\theta(n)}{q^n} + \left( \sum_{n=u_k-k+1}^{u_k-1} \frac{\theta(n)}{q^n} \right) + \frac{\theta(u_k)}{q^{u_k}} \\ &\qquad + \sum_{n=u_k+1}^{u_k+k} \frac{\theta(n)}{q^n} + \sum_{n=u_k+k+1}^{\infty} \frac{\theta(n)}{q^n} \\ &= \frac{r_k}{q^{u_k-k}} + \frac{\theta(u_k)}{q^{u_k}} + \sum_{n=u_k+k+1}^{\infty} \frac{\theta(n)}{q^n}, \end{aligned} \tag{17}$$

where $r_k$ is an integer. Multiplying both sides of (17) by $q^{u_k}$ and using Lemma 7, we obtain

$$|b_1 q^{u_k} - r_k q^k - \theta(u_k)| = \left| \frac{1}{q^k} \sum_{n=1}^{\infty} \frac{\theta(u_k + n + k)}{q^n} \right| \leq 2 \left( \frac{p}{|q|^\nu} \right)^{k/\nu}. \tag{18}$$

By the definition of $\nu$, the right-hand side in (18) tends to zero as $k$ tends to infnity, and so the integer

$$b_1 q^{u_k} + r_k q^k + \theta(u_k)$$

must be zero for sufficiently large $k$. Hence, $\theta(u_k)$ is a multiple of $q^k$ because

$$u_k = A_k i_k + B_k \geq A_k \geq q_1 > 4k^3 > k.$$

On the other hand, since $u_k \in T$,

$$\begin{aligned} |\theta(u_k)| &\leq M \sum_{\ell=2}^{m} a_\ell(u_k) \leq M \sum_{\ell=2}^{m} 2^{2^\ell} a p^{\frac{k}{\nu}} (1 + \log N_k)^{2^{\ell-1}} \\ &\leq 2^{2^m} a m M p^{\frac{k}{\nu}} (1 + \log N_k)^{2^{m-1}} \\ &< |q|^k. \end{aligned}$$

Therefore, $\theta(u_k) = 0$ for every large $k$ and Lemma 8 is proved. $\square$

## 4 Proof of Theorem 1

*Proof of Theorem* 1. Suppose on the contrary that the $m$ numbers given at (3)

$$1 \quad \text{and} \quad f_\ell(q^{-1}) = \sum_{n=1}^{\infty} a_\ell(n) q^{-n} \quad (\ell = 2, 3, \dots, m)$$

are linearly dependent over $\mathbb{Q}$. Then there exist integers $b_1$ and $b_\ell$ for $\ell = 2, 3, \dots, m$, not all zero, such that

$$b_1 \cdot 1 - \sum_{\ell=2}^{m} b_\ell f_\ell(q^{-1}) = 0,$$

and hence

$$b_1 = \sum_{\ell=2}^{m} b_\ell f_\ell(q^{-1}) = \sum_{n=1}^{\infty} \frac{\theta(n)}{q^n} \tag{19}$$

is an integer, where

$$\theta(n) := \sum_{\ell=2}^{m} b_\ell a_\ell(n).$$

Applying Lemma 8, we see that there exists $u_k \in T$ with $\theta(u_k) = 0$ for sufficiently large $k$.

On the other hand, the sequences $\{a_\ell(n)\}_{n\geq 1}$ $(\ell \geq 1)$ consist of nonnegative integers, and so we have

$$a_\ell(n) = \sum_{d|n} a_{\ell-1}(d) \geq a_{\ell-1}(n) \tag{20}$$

for every integer $n$. Furthermore, for each $\ell \geq 1$

$$a_\ell(u_k) \geq a_1(h) = a > 0, \tag{21}$$

since $u_k \in T_1$, so that $h \mid u_k$. Thus, by (20) and (21),

$$\begin{aligned} |\theta(u_k)| &= \left| \sum_{\ell=2}^{m} b_\ell a_\ell(u_k) \right| \\ &\geq |b_r a_r(u_k)| - \left| \sum_{\ell=2}^{r-1} b_\ell a_\ell(u_k) \right| \\ &\geq a_r(u_k) - M(r-2) \cdot a_{r-1}(u_k) \\ &= a_{r-1}(u_k) \left( \frac{a_r(u_k)}{a_{r-1}(u_k)} - mM \right), \end{aligned} \tag{22}$$

where $r \geq 2$ is the largest integer with $b_r \neq 0$. Since $u_k \in T_1$, the integer $u_k$ has the form $u_k = p^{\lambda_k}\eta_k$ with $\lambda_k \geq \lfloor k/(\nu+1) \rfloor$, where $p$ and $\eta_k$ are coprime. Hence, we have, by (7) and (20),

$$\frac{a_r(u_k)}{a_{r-1}(u_k)} = \left(1 + \frac{\lambda_k}{r-1}\right) \cdot \frac{a_h(\eta_k)}{a_{h-1}(\eta_k)} \geq 1 + \frac{\left\lfloor \frac{k}{\nu+1} \right\rfloor}{m-1} > mM$$

for all sufficiently large $k$, which implies that $\theta(u_k) \neq 0$ by (22). This is a contradiction which completes the proof of Theorem 1.

**Acknowledgments.**
The second author would like to express his gratitude to the conference organizers Professor Yuich Kamiya and Professor Hideaki Ishiokawa for giving him the possibility to give a talk in this pleasant conference. The present research was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B), 15K17504.

## References

[1] W. R. Alford, A. Granville, and C. Pomerance, There are infinitely many Carmichael numbers, *Ann. Math.* **140** (1994) 703–722.

[2] P. Erdős, On arithmetical properties of Lambert series, *J. Indian Math. Soc.* (N.S.) **12** (1948) 63–66.

[3] F. Luca and Y. Tachiya, Irrationality of Lambert series associated with a periodic sequence, *Int. J. Number Theory* **10** (2014), 623–636.

[4] M. B. Nathanson, Elementary methods in number theory, Graduate Texts inMathematics **195**, Springer-Verlag, New York, 2000.

F. Luca
School of Mathematics, University of the Witwatersrand,
Private Bag X3, Wits 2050, Johannesburg,
South Africa
e-mail: Florian.Luca@wits.ac.za

Y. Tachiya
Graduate School of Science and Technology, Hirosaki University,
Hirosaki 036–8561,
Japan
e-mail: tachiya@hirosaki-u.ac.jp